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Branes at $\mathbb{C}^4/\Gamma$ singularity from toric geometry

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Abstract: We study toric singularities of the form of $\mathbb{C}^4/\Gamma$ for finite abelian groups $\Gamma \subset \mathrm{SU}(4)$. In particular, we consider the simplest case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and find explicitly charge matrices for partial resolutions of this orbifold by extending the method by Morrison and Plesser. We obtain three kinds of algebraic equations, $z_1 z_2 z_3 z_4 = z_5^2, z_1 z_2 z_3 = z_4^2 z_5$ and $z_1 z_2 z_5 = z_3 z_4$ where z_i's parametrize $\mathbb{C}^5$. When we put N D1 branes at this singularity, it is known that the field theory on the worldvolume of N D1 branes is T-dual to $2 \times 2 \times 2$ brane cub model. We analyze geometric interpretation for field theory parameters and moduli space.

Keywords: D-branes, Supersymmetry and Duality, Field Theories in Lower Dimensions, Superstrings and Heterotic Strings.

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Contents

1. Introduction

In [1] the large N limit of superconformal field theories (SCFT) was described by taking the supergravity limit on anti-de Sitter (AdS) space. The scaling dimensions of operators of SCFT were obtained from the masses of particles in string/M theory [2]. In particular, $\mathcal{N} = 4$ SU(N) super Yang-Mills theory in 4 dimensions is described by Type IIB string theory on $AdS_5 \times \mathbf{S}^5$. This AdS/CFT correspondence can be tested by studying the Kaluza-Klein (KK) states of supergravity theory and by comparing them with the chiral primary operators of the SCFT on the boundary. There exist also $\mathcal{N} = 2, 1, 0$ conformal theories in 4 dimensions which have corresponding supergravity description on orbifolds [4, 5] of $AdS_5 \times \mathbf{S}^5$ [6].

On the other hand, T duality transforms orbifold singularity into NS5 branes and D3 branes into D4 branes. Brane box model [7] is connected by D3 branes at $\mathbb{C}^3/\Gamma$ singularity, through T duality. By extending this idea to three kinds of NS5 branes, it was found [8] that D1 branes, obtained from D4 branes by T duality, at $\mathbb{C}^4/\Gamma$ singularities with Γ an abelian subgroup of SU(4) are related to brane cub model. They discussed their method by comparing the result of [9] where $(0, 2)$ gauge theory on the worldvolume of D1 branes at the singular point of Calabi-Yau fourfold was studied.

In this paper, we generalize, the work of [10] where D branes are on the non orbifold singularities, to the case of toric singularities of $\mathbb{C}^4/\Gamma$ for finite abelian groups $\Gamma \subset$ SU(4) and we consider the simplest case $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. One of the singularities classified in [10] is so called conifold singularity which plays an important role in the D brane worldvolume theory. See the relevant papers [11, 12, 13, 14, 15]. In section 2, we find explicitly charge matrices for partial resolutions of this orbifold and obtain three kinds of algebraic equations. In section 3, we consider $2 \times 2 \times 2$ brane cub model which is dual to the field theory on the worldvolume of N D1 branes. In section 4, we study vacuum moduli space of D1 branes which is a toric variety. Finally in section 5, we will discuss important open problems and comment on the future directions.

2. Toric singularities

Before going to our present problems, we review the result of [10] in our context and see how it appears. If we let $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n$ act on $\mathbb{C}^3$ through the generators

$$\mathrm{diag}(e^{2\pi i/n}, e^{-2\pi i/n}, 1)\,, \qquad \mathrm{diag}(e^{2\pi i/n}, 1, e^{-2\pi i/n})\,, \tag{2.1}$$

then quotient singularity $\mathbb{C}^3/\Gamma$ is described by the polygons with vertices $(0,0)$, $(n,0)$ and $(0,n)$. The vectors consist of all integer vectors (k,l) with $k \geq 0, l \geq 0$ and $k+l \leq n$. In the simplest case of $\mathbb{Z}_2 \times \mathbb{Z}_2$, we label the vectors as follows [10]:

$$\begin{aligned} V_0 &= (0,0)\,, & V_1 &= (2,0)\,, & V_2 &= (0,2)\,, \\ W_0 &= (1,1)\,, & W_1 &= (0,1)\,, & W_2 &= (1,0)\,. \end{aligned} \tag{2.2}$$

Then the charge matrix in terms of homogeneous coordinates $X_0, X_1, X_2, Y_0, Y_1, Y_2$ becomes

$$\begin{pmatrix} X_0 & X_1 & X_2 & Y_0 & Y_1 & Y_2 \\ 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{pmatrix}. \tag{2.3}$$

The moment map is given by

$$\begin{aligned} |X_0|^2 + |Y_0|^2 - |Y_1|^2 - |Y_2|^2 &= \zeta_1\,, \\ |X_1|^2 - |Y_0|^2 + |Y_1|^2 - |Y_2|^2 &= \zeta_2\,, \\ |X_2|^2 - |Y_0|^2 - |Y_1|^2 + |Y_2|^2 &= \zeta_3\,. \end{aligned} \tag{2.4}$$

From now on we use slightly different approach unlike as [10] because our presentation will be more clear for the higher dimensional generalization and will lead to same result near the region of singularities. The vectors are

$$\begin{aligned} w_0 &= (0,0)\,, & v_1 &= (2,0)\,, & v_2 &= (0,2)\,, \\ u_0 &= (1,1)\,, & w_1 &= (1,0)\,, & w_2 &= (0,1)\,. \end{aligned} \tag{2.5}$$

From this, the charge matrix is given by and we denote its relation to those of [10] as follows:

$$\begin{pmatrix} t_0 = Y_0 & x_1 = X_1 & x_2 = X_2 & y_0 = X_0 & y_1 = Y_2 & y_2 = Y_1 \\ \hline 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 \end{pmatrix}. \tag{2.6}$$

Then the moment map is

$$\begin{aligned} |t_0|^2 + |y_0|^2 - |y_1|^2 - |y_2|^2 &= \eta_1 = \zeta_1 \,, \\ |x_1|^2 + |y_0|^2 - 2|y_1|^2 &= \eta_2 = \zeta_1 + \zeta_2 \,, \\ |x_2|^2 + |y_0|^2 - 2|y_2|^2 &= \eta_3 = \zeta_1 + \zeta_3 \,. \end{aligned} \tag{2.7}$$

Now it is easy to see that all four types classified by the variable ζ_i [10] are exactly the same as those by the variable η_i here: $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold corresponds to $\eta_1 = \eta_2 = \eta_3 = 0$, suspended pinch point to $\eta_1 = \eta_2 = 0$, conifold to $\eta_1 = 0$ and $\mathbb{Z}_2$ orbifold to $\eta_2 = 0$. Furthermore, $\eta_2 = \eta_3 = 0$ leads to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. Note that alternative description for this space were done in [17, 18].

Now we move on one higher dimensional case[1] and let $\Gamma = \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n$ act on $\mathbb{C}^4$ through the generators

$$\mathrm{diag}(e^{2\pi i/n}, e^{-2\pi i/n}, 1, 1) \,, \qquad \mathrm{diag}(e^{2\pi i/n}, 1, e^{-2\pi i/n}, 1) \qquad \mathrm{diag}(e^{2\pi i/n}, 1, 1, e^{-2\pi i/n}) \,, \tag{2.8}$$

then the quotient singularity $\mathbb{C}^4/\Gamma$ is described by the polygons with vertices

$$(0,0,0) \,, \qquad (n,0,0) \,, \qquad (0,n,0) \,, \qquad (0,0,n) \,. \tag{2.9}$$

A toric Gorenstein canonical singularity of complex dimension 4 is a convex polygon in $\mathbb{R}^3$ whose vertices have integer coordinates. Let all the vectors in $\mathbb{R}^3$ have integer coordinates and lie in either the interior or the boundary of the polygons. The set of vectors consist of all integer vectors (k,l,m) with $k \geq 0, l \geq 0, m \geq 0$ and $k+l+m \leq n$. In the case of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, let us label 10 vectors as follows:

$$\begin{aligned} &w_0 = (0,0,0) \,, \qquad v_1 = (2,0,0) \,, \qquad v_2 = (0,2,0) \,, \qquad v_3 = (0,0,2), \\ &w_1 = (1,0,0) \,, \qquad w_2 = (0,1,0) \,, \qquad w_3 = (0,0,1) \,, \\ &u_1 = (0,1,1) \,, \qquad u_2 = (1,0,1) \,, \qquad u_3 = (1,1,0) \,. \end{aligned} \tag{2.10}$$

[1]A generalization of the classical MaKay correspondence in complex dimension $d \geq 4$ requires the existence of projective crepant desingularizations. In [19], it was proved that all $(4;2)$ (in their notation) hyper surface singularities admit crepant projective desingularizations. We thank D. Dais for pointing out this.

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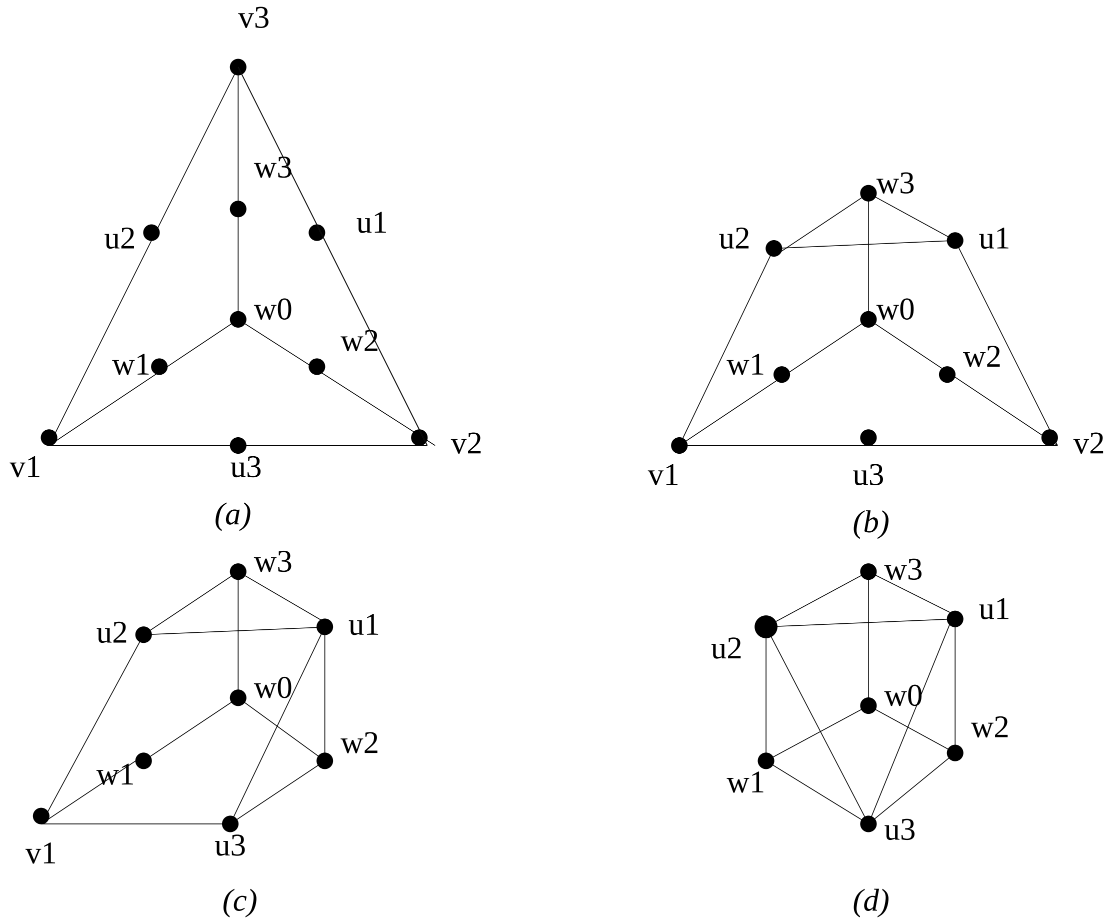

Figure 1: (a) Type I; (b) Type II-a; (c) Type III-a; (d) Type IV-a.

These vectors are drawn in fig 1a. The linear relations between these vectors with integer coefficients Q^j are

$$\sum_{j=1}^{3} Q^{j+1} v_j + \sum_{j=0}^{3} Q^{j+4} w_j + \sum_{j=1}^{3} Q^{j+7} u_j = 0 \tag{2.11}$$

satisfying the condition

$$\sum_{j=1}^{10} Q^j = 0\,. \tag{2.12}$$

Let $\vec{Q}_1, \ldots, \vec{Q}_6$ be a basis for the set of all such relations and use the matrix (Q_i^j) to specify the charges in a representation of $\mathrm{U}(1)^6$ on $\mathbb{C}^{10}$ where 6 equals the number of vectors minus the complex dimension (i.e. $10 - 4$). Then the singular space is the symplectic reduction $\mathbb{C}^{10}/\mathrm{U}(1)^6$. Partial and complete resolutions of this singularity can be obtained by changing the center of moment map.

By introducing homogeneous coordinates t_i, x_i, y_0, y_i, $(i = 1, 2, 3)$ and studying the $\mathrm{U}(1)^6$ action, the charge matrix is given by

$$Q_i^j = \left(\begin{array}{cccccccccc} u_1 & u_2 & u_3 & v_1 & v_2 & v_3 & w_0 & w_1 & w_2 & w_3 \\ \hline t_1 & t_2 & t_3 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \end{array} \right) . \tag{2.13}$$

Obviously the sum of the elements of each row vanishes according to (2.12). For convenience, we write the 10 vectors in the 1st row and homogeneous coordinates in the 2nd one. The moment map is

$$\begin{aligned} |t_1|^2 + |y_0|^2 - |y_2|^2 - |y_3|^2 &= \eta_1 \,, \\ |t_2|^2 + |y_0|^2 - |y_1|^2 - |y_3|^2 &= \eta_2 \,, \\ |t_3|^2 + |y_0|^2 - |y_1|^2 - |y_2|^2 &= \eta_3 \,, \\ |x_1|^2 + |y_0|^2 - 2|y_1|^2 &= \eta_4 \,, \\ |x_2|^2 + |y_0|^2 - 2|y_2|^2 &= \eta_5 \,, \\ |x_3|^2 + |y_0|^2 - 2|y_3|^2 &= \eta_6 \,. \end{aligned} \tag{2.14}$$

When η_i's are generic, the symplectic reduction is smooth. There are various singularity types for specific values of the η_i's. Some of these are illustrated in Fig1(b,c,d). Each singularity is related to a subgroup $\mathrm{U}(1)^k$ of $\mathrm{U}(1)^6$ such that at least $6-k$ of the homogeneous coordinates are uncharged under the $\mathrm{U}(1)^k$. The singular space is described by symplectic reduction of the space of $\mathbb{C}^{k+4}$ spanned by the remaining $k+4$ homogeneous coordinates by $\mathrm{U}(1)^k$. We list in table 1. and write the condition on D-term coefficients and the corresponding charge matrix on the remaining $k+4$ homogeneous coordinates. By starting from the case I) corresponding to zero of all η_i's and then move on the case in which five η_i's are vanishing: There are two cases whether the remaining nonzero η_j are either η_1, η_2, η_3 or η_4, η_5, η_6. Note that the charge matrix (2.13) is characterized by two parts: upper 3×10 matrix elements and lower 3×10 ones. So there are two cases on II): II-a) and II-b). The charge matrix for II-a) can be read off from I) by removing both 6th row and column. We can continue all the other cases and end up VI). Some of polygons corresponding to the relevant homogeneous coordinates are given in fig 1. As mentioned in the paper of [10], in our case also we can consider efficient description of the singularity possessing fewer fields and smaller group. But we do not pursue this issue here.

I) $\eta_i = 0 (i = 1, 2, \ldots, 6)$	$\begin{array}{cccccccccc} t_1 & t_2 & t_3 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \end{pmatrix}$
II-a) $\eta_i = 0 (i = 1, 2, 3, 4, 5)$	$\begin{array}{ccccccccc} t_1 & t_2 & t_3 & x_1 & x_2 & y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \end{pmatrix}$
II-b) $\eta_i = 0 (i = 1, 2, 4, 5, 6)$	$\begin{array}{ccccccccc} t_1 & t_2 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \end{pmatrix}$
III-a) $\eta_i = 0 (i = 1, 2, 3, 4)$	$\begin{array}{cccccccc} t_1 & t_2 & t_3 & x_1 & y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & -2 & 0 & 0 \end{pmatrix}$

Table 1: Charge matrices for partial resolutions of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and the conditions on D term coefficients. (Continues)

III-b) $\eta_1 = \eta_4 = \eta_5 = \eta_6 = 0$	$\begin{array}{c} \begin{matrix} t_1 & x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & -2 \end{pmatrix} \end{array}$
III-c) $\eta_1 = \eta_2 = \eta_4 = \eta_5 = 0$	$\begin{array}{c} \begin{matrix} t_1 & t_2 & x_1 & x_2 & y_0 & y_1 & y_2 & y_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & -2 & 0 \end{pmatrix} \end{array}$
IV-a) $\eta_1 = \eta_2 = \eta_3 = 0$	$\begin{array}{c} \begin{matrix} t_1 & t_2 & t_3 & y_0 & y_1 & y_2 & y_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & -1 & 0 \end{pmatrix} \end{array}$
IV-b) $\eta_1 = \eta_2 = \eta_4 = 0$	$\begin{array}{c} \begin{matrix} t_1 & t_2 & x_1 & y_0 & y_1 & y_2 & y_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -2 & 0 & 0 \end{pmatrix} \end{array}$
IV-c) $\eta_1 = \eta_4 = \eta_5 = 0$	$\begin{array}{c} \begin{matrix} t_1 & x_1 & x_2 & y_0 & y_1 & y_2 & y_3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -2 & 0 \end{pmatrix} \end{array}$

Table 1: Charge matrices for partial resolutions of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and the conditions on D term coefficients. (Continues)

IV-d) $\eta_4 = \eta_5 = \eta_6 = 0$	$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 \end{pmatrix} \end{array}$
V-a) $\eta_1 = \eta_2 = 0$	$\begin{array}{c} \begin{array}{cccccc} t_1 & t_2 & y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -1 & 0 & -1 \end{pmatrix} \end{array}$
V-b) $\eta_4 = \eta_5 = 0$	$\begin{array}{c} \begin{array}{ccccc} x_1 & x_2 & y_0 & y_1 & y_2 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & -2 \end{pmatrix} \end{array}$
V-c) $\eta_1 = \eta_4 = 0$	$\begin{array}{c} \begin{array}{cccccc} t_1 & x_1 & y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 1 & -2 & 0 & 0 \end{pmatrix} \end{array}$
VI-a) $\eta_1 = 0$	$\begin{array}{c} \begin{array}{ccccc} t_1 & y_0 & y_1 & y_2 & y_3 \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & -1 & -1 \end{pmatrix} \end{array}$
VI-b) $\eta_6 = 0$	$\begin{array}{c} \begin{array}{ccc} x_3 & y_0 & y_3 \end{array} \\ \begin{pmatrix} 1 & 1 & -2 \end{pmatrix} \end{array}$

Table 1: Charge matrices for partial resolutions of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold and the conditions on D term coefficients.

For each singularity, we can find $\mathrm{U}(1)^k$ invariant monomials and the equations they satisfy which give the algebraic relation of the singularity. They are given in table 2.

I)	$z_1 = y_0^2 y_1 y_2 y_3$, $z_3 = t_1 t_3 x_2^2 y_2$, $z_5 = t_1 t_2 t_3 x_1 x_2 x_3 y_0 y_1 y_2 y_3$	$z_2 = t_2 t_3 x_1^2 y_1$, $z_4 = t_1 t_2 x_3^2 y_3$	$z_1 z_2 z_3 z_4 = z_5^2 : (*)$

Table 2: Invariant monomials and equations for partial resolutions of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. We denote three different kinds of algebraic equations as $(*), (**)$ and $(***)$ inside the last column. (Continues)

II-a)	$z_1 = y_0^2 y_1 y_2 y_3,\ z_2 = t_2 t_3 x_1^2 y_1,$ $z_3 = t_1 t_3 x_2^2 y_2,\ z_4 = t_3 x_1 x_2 y_0 y_1 y_2,$ $z_5 = t_1 t_2 y_3$	$z_1 z_2 z_3 = z_4^2 z_5 : (**)$
II-b)	$z_1 = y_0^2 y_1 y_2 y_3,\quad z_2 = t_1 t_2 x_3^2 y_3,$ $z_3 = t_1 x_2^2 y_2,\quad z_4 = t_2 x_1^2 y_1,$ $z_5 = t_1 t_2 x_1 x_2 x_3 y_0 y_1 y_2 y_3$	$z_1 z_2 z_3 z_4 = z_5^2 : (*)$
III-a)	$z_1 = y_0^2 y_1 y_2 y_3,\quad z_2 = t_2 t_3 x_1^2 y_1,$ $z_3 = t_1 t_2^2 x_1 y_0 y_1 y_3^2,\ z_4 = t_3 x_1 y_0 y_1 y_2,$ $z_5 = t_1 t_2 y_3$	$z_1 z_2 z_5 = z_3 z_4 : (***)$
III-b)	$z_1 = x_1^2 y_1,\quad z_2 = y_0^2 y_1 y_2 y_3,$ $z_3 = t_1 x_2^2 y_2,\quad z_4 = t_1 x_3^2 y_3,$ $z_5 = t_1 x_1 x_2 x_3 y_0 y_1 y_2 y_3$	$z_1 z_2 z_3 z_4 = z_5^2 : (*)$
III-c)	$z_1 = y_0^2 y_1 y_2 y_3,\ z_2 = t_2 x_1^2 y_1,$ $z_3 = t_1 x_2^2 y_2,\quad z_4 = x_1 x_2 y_0 y_1 y_2,$ $z_5 = t_1 t_2 y_3$	$z_1 z_2 z_3 = z_4^2 z_5 : (**)$
IV-a)	$z_1 = t_2 y_0 y_1 y_3,\ z_2 = t_1 y_0 y_2 y_3,$ $z_3 = y_0^2 y_1 y_2 y_3,\ z_4 = t_1 t_2 t_3 y_0 y_1 y_2 y_3,$ $z_5 = t_3 y_0 y_1 y_2$	$z_1 z_2 z_5 = z_3 z_4 : (***)$
IV-b)	$z_1 = t_1 y_2,\quad z_2 = y_0^2 y_1 y_2 y_3,$ $z_3 = t_2 x_1^2 y_1,\ z_4 = x_1 y_0 y_1 y_2,$ $z_5 = t_1 t_2 y_3$	$z_1 z_2 z_3 = z_4^2 z_5 : (**)$
IV-c)	$z_1 = t_1 x_2^2 y_2,\ z_2 = y_0^2 y_1 y_2 y_3,$ $z_3 = x_1^2 y_1,\quad z_4 = x_1 x_2 y_0 y_1 y_2,$ $z_5 = t_1 y_3$	$z_1 z_2 z_3 = z_4^2 z_5 : (**)$

Table 2: Invariant monomials and equations for partial resolutions of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. We denote three different kinds of algebraic equations as $(*), (**)$ and $(***)$ inside the last column. (Continues)

IV-d)	$z_1 = x_3^2 y_3, \quad z_2 = y_0^2 y_1 y_2 y_3,$ $z_3 = x_1^2 y_1, \quad z_4 = x_2^2 y_2,$ $z_5 = x_1 x_2 x_3 y_0 y_1 y_2 y_3$	$z_1 z_2 z_3 z_4 = z_5^2 : (*)$
V-a)	$z_1 = y_0 y_3, \quad z_2 = t_1 y_2,$ $z_3 = y_0 y_1 y_2,\ z_4 = t_1 t_2 y_3,$ $z_5 = t_2 y_1$	$z_1 z_2 z_5 = z_3 z_4 : (***)$
V-b)	$z_1 = y_0^2 y_1 y_2,\ z_2 = x_1^2 y_1,$ $z_3 = x_2^2 y_2, \quad z_4 = x_1 x_2 y_0 y_1 y_2,$	$z_1 z_2 z_3 = z_4^2$
V-c)	$z_1 = t_1 y_3, \quad z_2 = y_0^2 y_1 y_2 y_3,$ $z_3 = t_1 y_2, \quad z_4 = x_1^2 y_1,$ $z_5 = t_1 x_1 y_0 y_1 y_2 y_3$	$z_1 z_2 z_3 z_4 = z_5^2 : (*)$
VI-a)	$z_1 = t_1 y_2,\ z_2 = t_1 y_3,$ $z_3 = y_0 y_2,\ z_4 = y_0 y_3,$	$z_1 z_4 = z_2 z_3$
VI-b)	$z_1 = x_3^2 y_3,\ z_2 = x_3 y_0 y_3,$ $z_3 = y_0^2 y_3,$	$z_1 z_3 = z_2^2$

Table 2: Invariant monomials and equations for partial resolutions of the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold. We denote three different kinds of algebraic equations as $(*), (**)$ and $(***)$ inside the last column.

The presentation of these singularities includes all possible toric blown ups by varying the D terms. The remaining three cases of V-b), VI-a) and VI-b) correspond to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold, conifold, and $\mathbb{Z}_2$ orbifold, for the quotient singularity $\mathbb{C}^3/\Gamma$ respectively studied in [10].

3. Branes at a $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ singularity

Brane configurations [8] of NS5, NS5', NS5" and D4 branes in Type IIA string theory are

$$\begin{aligned}
&2 \text{ NS5 branes along } (x^0, x^1, x^2, x^3, x^4, x^5), \\
&2 \text{ NS5' branes along } (x^0, x^1, x^2, x^3, x^6, x^7), \\
&2 \text{ NS5" branes along } (x^0, x^1, x^4, x^5, x^6, x^7), \\
&N \text{ D4 branes along } (x^0, x^1, x^2, x^4, x^6)
\end{aligned} \tag{3.1}$$

JHEP04(1999)012

where D4 branes are finite in the direction (x^2, x^4, x^6). They are bounded in the direction x^2 by the NS5" branes, in the direction x^4 by the NS5' branes, and in the direction x^6 by the NS5 branes. The coordinates of all branes in (x^8, x^9) should be equal, 2 NS5 branes should have the same position in x^7, 2 NS5' branes should have the same position in x^5 and 2 NS5" branes should have the same position in x^3. The low energy effective field theory on the D4 branes is $(0, 2)$ supersymmetric theory in 2 dimension because (x^0, x^1) are the only noncompact directions in their worldvolume. The existence of each kind of NS5 brane breaks one half of the supersymmetries and breaks to 1/8 of the original supersymmetry. A further half is broken by the presence of D4 branes and the worldvolume theory therfore is $(0, 2)$ supersymmetric theory in 2 dimension. The $\mathrm{U}(1)_R$ R symmetry of the field theory is the rotational symmetry in (x^8, x^9) directions. Let us perform T duality on the brane cub model along the directions (x^2, x^4, x^6). The T duality along x^2 direction transforms the NS5" branes along $(x^0, x^1, x^4, x^5, x^6, x^7)$ into 2 KK" monopoles. This gives singularities of type A_1. The T duality along the x^4 direction transforms NS5' branes along $(x^0, x^1, x^2, x^3, x^6, x^7)$ into 2 KK' monopoles which produce type A_1 singularity. Finally the T duality along the x^6 direction transforms NS5 branes along $(x^0, x^1, x^2, x^3, x^4, x^5)$ into 2 KK monopoles which will be A_1 singularity. Then the final T dual of NS5, NS5' and NS5" branes is Type IIB string theory with a complicated geometry in the directions $(x^{2'}, x^3, x^{4'}, x^5, x^{6'}, x^7, x^8, x^9)$. Each complex surface of A_1 singularity is characterized by two equations respectively and at the origin all surfaces meet. This can be described by a quotient singularity of type $\mathbb{C}^4/\Gamma$ with $\Gamma = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. After T duality the original D4 branes become D1 branes located at the singular point. Since the original D4 branes are bounded by the grid of NS5, NS5' and NS5" branes, the T dual D1 branes will be located exactly at the $\mathbb{C}^4/\Gamma$ singularity.

The field theory on the worldvolume of N D1 branes on $\mathbb{C}^4$ can be obtained from ten dimensional $\mathcal{N} = 1$ $\mathrm{U}(N)$ super Yang-Mills by dimensional reduction. Then the field theory of $D1$ branes on $\mathbb{C}^4/\Gamma$ can be obtained by a projection into Γ invariant states. The group Γ acts on the R symmetry of the theory and is a subgroup of $\mathrm{SU}(4)$. An unbroken $\mathrm{U}(1)_R$ R symmetry will be present in the quotient theory due to the decomposition $\mathrm{SU}(4) \times \mathrm{U}(1)_R \subset \mathrm{SO}(8)_R$. The action of Γ should be embedded in the Chan-Paton indices of D1 branes. The fields in the resulting $(0, 2)$ field theory on the D1 branes are those invariant under both R symmetry and gauge quantum numbers. Following the rule of [4], the spectrum can be obtained. There exist four kinds of complex scalar fields $\Phi_{I,I\oplus A_i}, i = 1, 2, 3, 4$ according to the tensor products of the representation $\mathbf{4}$ with each irreducible representation. The fields associated to the first complex plane $\Phi^1_{a,b,c}$ transform in the $(\square, \bar{\square})$ of $\mathrm{U}(N)_{a,b,c} \times \mathrm{U}(N)_{a+1,b,c}$. The fields $\Phi^2_{a,b,c}$ transform in the $(\square, \bar{\square})$ of $\mathrm{U}(N)_{a,b,c} \times \mathrm{U}(N)_{a,b+1,c}$. The fields $\Phi^3_{a,b,c}$ transform in the $(\square, \bar{\square})$ of $\mathrm{U}(N)_{a,b,c} \times \mathrm{U}(N)_{a,b,c+1}$. The last one $\Phi^4_{a,b,c}$ transforms in the $(\square, \bar{\square})$ of $\mathrm{U}(N)_{a,b,c} \times \mathrm{U}(N)_{a-1,b-1,c-1}$. The chiral multiplets are listed in table 3 and the corresponding Fermi multiplets are given in table 4.

Field	Representations under $\mathrm{U}(N)_{1,1,1} \times \mathrm{U}(N)_{2,1,1} \times \mathrm{U}(N)_{1,2,1} \times \mathrm{U}(N)_{2,2,1}$ $\times \mathrm{U}(N)_{1,1,2} \times \mathrm{U}(N)_{2,1,2} \times \mathrm{U}(N)_{1,2,2} \times \mathrm{U}(N)_{2,2,2}$
$\Phi^1_{1,1,1}; \overline{\Phi^1_{2,1,1}}$	$(\Box, \bar{\Box}, 1, 1, 1, 1, 1, 1)$
$\Phi^1_{1,2,1}; \overline{\Phi^1_{2,2,1}}$	$(1, 1, \Box, \bar{\Box}, 1, 1, 1, 1)$
$\Phi^1_{1,1,2}; \overline{\Phi^1_{2,1,2}}$	$(1, 1, 1, 1, \Box, \bar{\Box}, 1, 1)$
$\Phi^1_{1,2,2}; \overline{\Phi^1_{2,2,2}}$	$(1, 1, 1, 1, 1, 1, \Box, \bar{\Box})$
$\Phi^2_{1,1,1}; \overline{\Phi^2_{1,2,1}}$	$(\Box, 1, \bar{\Box}, 1, 1, 1, 1, 1)$
$\Phi^2_{2,1,1}; \overline{\Phi^2_{2,2,1}}$	$(1, \Box, 1, \bar{\Box}, 1, 1, 1, 1)$
$\Phi^2_{1,1,2}; \overline{\Phi^2_{1,2,2}}$	$(1, 1, 1, 1, \Box, 1, \bar{\Box}, 1)$
$\Phi^2_{2,1,2}; \overline{\Phi^2_{2,2,2}}$	$(1, 1, 1, 1, 1, \Box, 1, \bar{\Box})$
$\Phi^3_{1,1,1}; \overline{\Phi^3_{1,1,2}}$	$(\Box, 1, 1, 1, \bar{\Box}, 1, 1, 1)$
$\Phi^3_{2,1,1}; \overline{\Phi^3_{2,1,2}}$	$(1, \Box, 1, 1, 1, \bar{\Box}, 1, 1)$
$\Phi^3_{1,2,1}; \overline{\Phi^3_{1,2,2}}$	$(1, 1, \Box, 1, 1, 1, \bar{\Box}, 1)$
$\Phi^3_{2,2,1}; \overline{\Phi^1_{2,2,2}}$	$(1, 1, 1, \Box, 1, 1, 1, \bar{\Box})$
$\Phi^4_{1,1,1}; \overline{\Phi^4_{2,2,2}}$	$(\Box, 1, 1, 1, 1, 1, 1, \bar{\Box})$
$\Phi^4_{2,1,1}; \overline{\Phi^4_{1,2,2}}$	$(1, \Box, 1, 1, 1, 1, \bar{\Box}, 1)$
$\Phi^4_{1,2,1}; \overline{\Phi^4_{2,1,2}}$	$(1, 1, \Box, 1, 1, \bar{\Box}, 1, 1)$
$\Phi^4_{2,2,1}; \overline{\Phi^4_{1,1,2}}$	$(1, 1, 1, \Box, \bar{\Box}, 1, 1, 1)$

Table 3: Chiral multiplets for the theory of D1 branes at singularities. Obviously $\Phi^1_{2,1,1}$ is a conjugate representation of $\Phi^1_{1,1,1}$ and $\overline{\Phi^1_{1,1,1}}$ is a conjugate representation of $\overline{\Phi^1_{2,1,1}}$ and so on.

The superpotential [9] is written, by taking the field $\Phi^4_{a,b,c}$ as special one, in terms of brane cub model description

$$\begin{aligned} W = \sum_{a,b,c=1}^{2} \Big[& \Lambda^{24}_{a+1,b,c+1} \left(\Phi^1_{a,b,c} \Phi^3_{a+1,b,c} - \Phi^3_{a,b,c} \Phi^1_{a,b,c+1} \right) + \\ & + \Lambda^{14}_{a,b+1,c+1} \left(\Phi^3_{a,b,c} \Phi^2_{a,b,c+1} - \Phi^2_{a,b,c} \Phi^3_{a,b+1,c} \right) + \\ & + \Lambda^{34}_{a+1,b+1,c} \left(\Phi^2_{a,b,c} \Phi^1_{a,b+1,c} - \Phi^1_{a,b,c} \Phi^2_{a+1,b,c} \right) \Big] . \end{aligned} \tag{3.2}$$

Field	Representations under $\mathrm{U}(N)_{1,1,1} \times \mathrm{U}(N)_{2,1,1} \times \mathrm{U}(N)_{1,2,1} \times \mathrm{U}(N)_{2,2,1}$ $\times \mathrm{U}(N)_{1,1,2} \times \mathrm{U}(N)_{2,1,2} \times \mathrm{U}(N)_{1,2,2} \times \mathrm{U}(N)_{2,2,2}$
$\Lambda^{24}_{1,1,1}; \overline{\Lambda^{24}_{2,1,2}}$	$(\Box, 1, 1, 1, 1, \bar{\Box}, 1, 1)$
$\Lambda^{24}_{2,1,1}; \overline{\Lambda^{24}_{1,1,2}}$	$(1, \Box, 1, 1, \bar{\Box}, 1, 1, 1)$
$\Lambda^{24}_{1,2,1}; \overline{\Lambda^{24}_{2,2,2}}$	$(1, 1, \Box, 1, 1, 1, 1, \bar{\Box})$
$\Lambda^{24}_{2,2,1}; \overline{\Lambda^{24}_{1,2,2}}$	$(1, 1, 1, \Box, 1, 1, \bar{\Box}, 1)$
$\Lambda^{14}_{1,1,1}; \overline{\Lambda^{14}_{1,2,2}}$	$(\Box, 1, 1, 1, 1, 1, \bar{\Box}, 1)$
$\Lambda^{14}_{1,2,1}; \overline{\Lambda^{14}_{1,1,2}}$	$(1, 1, \Box, 1, \bar{\Box}, 1, 1, 1)$
$\Lambda^{14}_{2,1,1}; \overline{\Lambda^{14}_{2,2,2}}$	$(1, \Box, 1, 1, 1, 1, 1, \bar{\Box})$
$\Lambda^{14}_{2,2,1}; \overline{\Lambda^{14}_{2,1,2}}$	$(1, 1, 1, \Box, 1, \bar{\Box}, 1, 1)$
$\Lambda^{34}_{1,1,1}; \overline{\Lambda^{34}_{2,2,1}}$	$(\Box, 1, 1, \bar{\Box}, 1, 1, 1, 1)$
$\Lambda^{34}_{2,1,1}; \overline{\Lambda^{34}_{1,2,1}}$	$(1, \Box, \bar{\Box}, 1, 1, 1, 1, 1)$
$\Lambda^{34}_{1,1,2}; \overline{\Lambda^{34}_{2,2,2}}$	$(1, 1, 1, 1, \Box, 1, 1, \bar{\Box})$
$\Lambda^{34}_{2,1,2}; \overline{\Lambda^{34}_{1,2,2}}$	$(1, 1, 1, 1, 1, \Box, \bar{\Box}, 1)$

Table 4: Fermi multiplets for the the theory of D1 branes at singularities. $\Lambda^{24}_{2,1,2}$ is a conjugate representation of $\Lambda^{24}_{1,1,1}$ and $\overline{\Lambda^{24}_{1,1,1}}$ is a conjugate representation of $\overline{\Lambda^{24}_{2,1,2}}$ and so on.

Here $\Phi^1_{a,b,c}$ is represented by an arrow which goes from the box (a, b, c) to the box $(a+1, b, c)$. $\Phi^3_{a+1,b,c}$ is represented by an arrow which goes from the box $(a+1, b, c)$ to the box $(a+1, b, c+1)$. Furthermore $\Lambda^{24}_{a+1,b,c+1}$ is represented by an arrow which goes from the box $(a+1, b, c+1)$ to the box $(a+2, b, c+2)$. So one can see the first term of (3.2) is a gauge invariant object because the final box $(a+2, b, c+2)$ is nothing but the initial box (a, b, c) and they form a closed oriented triangle. We can check all the other terms in (3.2) form a closed triangles and the superpotential W is a gauge invariant quantity. The D term equations for this case are

$$\sum_{i=1}^{4} |\Phi^i_{1,1,1}|^2 - |\Phi^1_{2,1,1}|^2 - |\Phi^2_{1,2,1}|^2 - |\Phi^3_{1,1,2}|^2 - |\Phi^4_{2,2,2}|^2 = r_{1,1,1}\,,$$
$$\sum_{i=1}^{4} |\Phi^i_{1,2,1}|^2 - |\Phi^1_{2,2,1}|^2 - |\Phi^2_{1,2,1}|^2 - |\Phi^3_{1,2,2}|^2 - |\Phi^4_{2,1,2}|^2 = r_{1,2,1}\,,$$

$$\sum_{i=1}^{4} |\Phi^i_{1,1,2}|^2 - |\Phi^1_{2,1,2}|^2 - |\Phi^2_{1,2,2}|^2 - |\Phi^3_{1,1,1}|^2 - |\Phi^4_{2,2,1}|^2 = r_{1,1,2}\,,$$
$$\sum_{i=1}^{4} |\Phi^i_{1,2,2}|^2 - |\Phi^1_{2,2,2}|^2 - |\Phi^2_{1,1,2}|^2 - |\Phi^3_{1,2,1}|^2 - |\Phi^4_{2,1,1}|^2 = r_{1,2,2}\,,$$
$$\sum_{i=1}^{4} |\Phi^i_{2,1,1}|^2 - |\Phi^1_{1,1,1}|^2 - |\Phi^2_{2,2,1}|^2 - |\Phi^3_{2,1,2}|^2 - |\Phi^4_{1,2,2}|^2 = r_{2,1,1}\,,$$
$$\sum_{i=1}^{4} |\Phi^i_{2,2,1}|^2 - |\Phi^1_{1,2,1}|^2 - |\Phi^2_{2,1,1}|^2 - |\Phi^3_{2,2,2}|^2 - |\Phi^4_{2,2,1}|^2 = r_{2,2,1}\,,$$
$$\sum_{i=1}^{4} |\Phi^i_{2,1,2}|^2 - |\Phi^1_{1,1,2}|^2 - |\Phi^2_{2,2,2}|^2 - |\Phi^3_{2,1,1}|^2 - |\Phi^4_{2,1,2}|^2 = r_{2,1,2}\,,$$
$$\sum_{i=1}^{4} |\Phi^i_{2,2,2}|^2 - |\Phi^1_{1,2,2}|^2 - |\Phi^2_{2,1,2}|^2 - |\Phi^3_{2,2,1}|^2 - |\Phi^4_{1,1,1}|^2 = r_{2,2,2}\,, \tag{3.3}$$

where Fayet-Iliopoulos parameter $r_{a,b,c} = (x^7)_{NS_a} - (x^7)_{NS_{a+1}} + (x^5)_{NS_b} - (x^5)_{NS_{b+1}} + (x^3)_{NS_c} - (x^3)_{NS_{c+1}} = (\delta x^7)_a + (\delta x^5)_b + (\delta x^3)_c$ and satisfies the condition $\sum_{a,b,c=1}^{2} r_{a,b,c} = 0$, which is evident from the above expression, in order to have unbroken supersymmetry. They also satisfy F term equations given by

$$\begin{aligned}
\Phi^1_{a,b,c}\Phi^3_{a+1,b,c} &= \Phi^3_{a,b,c}\Phi^1_{a,b,c+1}\,,\\
\Phi^3_{a,b,c}\Phi^2_{a,b+1,c} &= \Phi^2_{a,b,c}\Phi^3_{a,b+1,c}\,,\\
\Phi^2_{a,b,c}\Phi^1_{a,b+1,c} &= \Phi^1_{a,b,c}\Phi^2_{a+1,b,c}\,,\\
\Phi^4_{a+1,b,c+1}\Phi^2_{a,b-1,c} &= \Phi^2_{a+1,b,c+1}\Phi^4_{a+1,b+1,c+1}\,,\\
\Phi^4_{a,b+1,c+1}\Phi^1_{a-1,b,c} &= \Phi^1_{a,b+1,c+1}\Phi^4_{a+1,b+1,c+1}\,,\\
\Phi^4_{a+1,b+1,c}\Phi^3_{a,b,c-1} &= \Phi^3_{a+1,b+1,c}\Phi^4_{a+1,b+1,c+1}\,.
\end{aligned} \tag{3.4}$$

The first three conditions can be read off from the Fermi superfields in the form of superpotential in (3.2). The remaining three conditions come from other types of triangles.

4. D-term equations and moduli space

Now we take the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ action generators g_1, g_2 and g_3 to act on $\mathbb{C}^4$ as follows:

$$\begin{aligned}
g_1 &: (X,Y,Z,W) \to (-X,-Y,Z,W)\,,\\
g_2 &: (X,Y,Z,W) \to (-X,Y,-Z,W)\,,\\
g_3 &: (X,Y,Z,W) \to (-X,Y,Z,-W)\,.
\end{aligned} \tag{4.1}$$

After diagonalizing, the regular representations of Γ are given by

$$
\begin{aligned}
S(g_1) &= \mathrm{diag}(1,1,1,1,-1,-1,-1,-1)\,,\\
S(g_2) &= \mathrm{diag}(1,1,-1,-1,1,1,-1,-1)\,,\\
S(g_3) &= \mathrm{diag}(1,-1,1,-1,1,-1,1,-1)\,.
\end{aligned} \tag{4.2}
$$

Let X, Y, Z and W be the 8×8 matrices from Yang-Mills theory. They are the lowest components of superfield $\Phi^i_{a,b,c}(i=1,2,3,4)$ respectively and have the constraints

$$
\begin{array}{lll}
X = -S(g_1)XS(g_1)^{-1}\,, & X = -S(g_2)XS(g_2)^{-1}\,, & X = -S(g_3)XS(g_3)^{-1}\,,\\
Y = -S(g_1)YS(g_1)^{-1}\,, & Y = S(g_2)YS(g_2)^{-1}\,, & Y = S(g_3)YS(g_3)^{-1}\,,\\
Z = S(g_1)ZS(g_1)^{-1}\,, & Z = -S(g_2)ZS(g_2)^{-1}\,, & Z = S(g_3)ZS(g_3)^{-1}\,,\\
W = S(g_1)WS(g_1)^{-1}\,, & W = S(g_2)WS(g_2)^{-1}\,, & W = -S(g_3)WS(g_3)^{-1}\,.
\end{array} \tag{4.3}
$$

Then the surviving fields after projections are

$$
\begin{aligned}
(x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8) &= (X_{18},X_{27},X_{36},X_{45},X_{54},X_{63},X_{72},X_{81})\\
(y_1,y_2,y_3,y_4,y_5,y_6,y_7,y_8) &= (Y_{15},Y_{26},Y_{37},Y_{48},Y_{51},Y_{62},Y_{73},Y_{84})\\
(z_1,z_2,z_3,z_4,z_5,z_6,z_7,z_8) &= (Z_{13},Z_{24},Z_{31},Z_{42},Z_{57},Z_{68},Z_{75},Z_{86})\\
(w_1,w_2,w_3,w_4,w_5,w_6,w_7,w_8) &= (W_{12},W_{21},W_{34},W_{43},W_{56},W_{65},W_{78},W_{87}).
\end{aligned} \tag{4.4}
$$

From the F-flatness conditions corresponding to (3.4)

$$
\begin{array}{lll}
[X,Y]=0\,, & [X,Z]=0\,, & [X,W]=0\,,\\
[Y,Z]=0\,, & [Y,W]=0\,, & [Z,W]=0\,,
\end{array} \tag{4.5}
$$

the independent variables are as follows

$$
x_1,x_2,x_3,x_4,x_5,y_1,y_2,y_3,z_1,z_2,w_1\,. \tag{4.6}
$$

That is 11 dimensional affine toric variety. Note that $11-(8-1)=4$. The cone is generated by the rows of the rectangular 32×11 matrix. The remaining variables are expressed in terms of these 11 independent variables. In usual way, we write them as follows

	x_1	x_2	x_3	x_4	x_5	y_1	y_2	y_3	z_1	z_2	w_1
x_1	1	0	0	0	0	0	0	0	0	0	0
x_2	0	1	0	0	0	0	0	0	0	0	0
x_3	0	0	1	0	0	0	0	0	0	0	0
x_4	0	0	0	1	0	0	0	0	0	0	0
x_5	0	0	0	0	1	0	0	0	0	0	0
x_6	0	0	-1	1	1	0	0	0	0	0	0
x_7	0	-1	0	1	1	0	0	0	0	0	0
x_8	-1	0	0	1	1	0	0	0	0	0	0
y_1	0	0	0	0	0	1	0	0	0	0	0
y_2	0	0	0	0	0	0	1	0	0	0	0
y_3	0	0	0	0	0	0	0	1	0	0	0
y_4	1	-1	-1	1	0	-1	1	1	0	0	0
y_5	0	-1	-1	1	1	-1	1	1	0	0	0
y_6	0	-1	-1	1	1	0	0	1	0	0	0
y_7	0	-1	-1	1	1	0	1	0	0	0	0
y_8	-1	0	0	0	1	1	0	0	0	0	0
z_1	0	0	0	0	0	0	0	0	1	0	0
z_2	0	0	0	0	0	0	0	0	0	1	0
z_3	0	-1	0	1	0	-1	0	1	0	1	0
z_4	0	-1	0	1	0	-1	0	1	1	0	0
z_5	0	0	0	0	0	-1	0	1	1	0	0
z_6	1	-1	-1	1	0	-1	0	1	0	1	0
z_7	0	-1	0	1	0	0	0	0	0	1	0
z_8	-1	0	1	0	0	0	0	0	1	0	0
w_1	0	0	0	0	0	0	0	0	0	0	1
w_2	0	0	-1	1	0	-1	1	0	-1	1	1
w_3	0	0	0	0	0	0	0	0	-1	1	1
w_4	0	0	-1	1	0	-1	1	0	0	0	1
w_5	0	0	0	0	0	-1	1	0	0	0	1
w_6	0	0	-1	1	0	0	0	0	-1	1	1
w_7	1	-1	-1	1	0	-1	1	0	-1	1	1
w_8	-1	1	0	0	0	0	0	0	0	0	1

In the above, for example, 6 row implies $x_6 = x_4 x_5 / x_3$ and others hold similarly.

In order to describe toric variety it is convenient to consider dual cone. The primitive generators[2] $v_1, \ldots, v_{34} \subset \mathbb{Z}^{11}$ define the linear map $T : \mathbb{Z}^{34} \to \mathbb{N} = \mathbb{Z}^{11}$ which generates the dual cone. The inner product between 32 vectors in (5) and $v_i (i = 1, \ldots, 34)$ are greater than zero.

$$
\begin{aligned}
v_1 &= (0,0,0,1,0,0,0,0,0,0,0)\,, & v_2 &= (0,0,0,0,1,0,0,0,0,0,0)\,,\\
v_3 &= (0,0,0,0,0,0,1,0,0,0,0)\,, & v_4 &= (0,0,0,0,0,0,0,1,0,0,0)\,,\\
v_5 &= (0,0,0,0,0,0,0,0,0,1,0)\,, & v_6 &= (0,0,0,0,0,0,0,0,0,0,1)\,,\\
v_7 &= (0,1,0,1,0,0,0,0,0,0,0)\,, & v_8 &= (0,0,1,1,0,0,0,0,0,0,0)\,,\\
v_9 &= (0,0,0,0,0,0,0,0,1,1,0)\,, & v_{10} &= (0,0,0,0,0,0,0,0,1,0,1)\,,\\
v_{11} &= (0,0,0,0,0,1,1,1,0,0,0)\,, & v_{12} &= (0,0,0,0,0,1,0,1,0,0,1)\,,\\
v_{13} &= (1,0,1,0,1,0,0,0,0,0,1)\,, & v_{14} &= (1,0,0,0,1,0,0,0,1,0,1)\,,\\
v_{15} &= (0,1,0,0,1,0,0,1,0,1,0)\,, & v_{16} &= (0,0,1,0,1,0,1,0,0,1,0)\,,\\
v_{17} &= (0,0,1,0,1,0,0,1,0,0,1)\,, & v_{18} &= (0,0,0,1,0,1,0,0,1,0,1)\,,\\
v_{19} &= (0,0,0,0,0,1,1,0,1,1,0)\,, & v_{20} &= (1,1,1,1,1,0,0,0,0,0,0)\,,\\
v_{21} &= (1,1,0,0,1,0,0,0,1,1,0)\,, & v_{22} &= (1,0,0,1,0,1,0,0,1,0,1)\,,\\
v_{23} &= (0,1,1,1,0,0,1,1,0,0,0)\,, & v_{24} &= (0,1,0,0,1,0,1,0,1,1,0)\,,\\
v_{25} &= (1,0,1,1,0,1,0,1,0,0,1)\,, & v_{26} &= (1,0,0,0,1,1,0,0,1,1,1)\,,\\
v_{27} &= (0,1,1,0,1,0,1,1,0,1,0)\,, & v_{28} &= (1,1,1,1,0,1,1,1,0,0,0)\,,\\
v_{29} &= (1,1,1,0,1,0,1,1,0,1,0)\,, & v_{30} &= (1,1,0,1,0,1,1,0,1,1,0)\,,\\
v_{31} &= (1,1,0,1,0,1,0,1,1,0,1)\,, & v_{32} &= (1,0,1,1,0,1,1,0,1,0,1)\,,\\
v_{33} &= (0,1,1,1,0,1,1,1,0,1,0)\,, & v_{34} &= (1,1,1,0,1,1,1,1,1,1,1)\,.
\end{aligned}
\tag{4.7}
$$

Also we can get (5) from these primitive generators. The linear map T has the matrix elements $T = (v_1^t, \ldots, v_{34}^t)$ where t is a transpose operation. The kernel of T matrix,

[2]The explicit calculation of these is due to K. Mohri who used PORTA, c-program of POlyhedron Representation Transformation Algorithm which can be obtained from the site, `ftp://ftp.zib.de/pub/Packages/mathprog/polyth/index.html`.

which is 23×34 matrix and denoted by Q_F, is given by

$$
Q_F = \left(\begin{array}{ccccccccccccc|c}
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & \mathbf{0_{1\times 21}} \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \mathbf{0_{1\times 21}} \\
\hline
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & \\
1 & -1 & 0 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\
1 & -1 & -1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \\
1 & -1 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \\
-1 & 0 & 1 & 1 & 1 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & -1 & \\
-2 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & \mathbf{1_{21\times 21}} \\
1 & 0 & -1 & -1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & \\
1 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & \\
-1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & -1 & \\
2 & -1 & -1 & -1 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & \\
1 & 0 & -1 & -1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & \\
-1 & 1 & 0 & 1 & 0 & 1 & -1 & 1 & -1 & 0 & -1 & 0 & -1 & \\
-1 & 1 & 1 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & -1 & 0 & -1 & \\
-1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & \\
1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & -1 & 0 & 0 & \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 & 0 & -1 &
\end{array}\right). \qquad (4.8)
$$

Here we denote 1×21 zero matrix by $\mathbf{0_{1\times 21}}$ and 21×21 identity matrix by $\mathbf{1_{21\times 21}}$ for simplicity. Q_F satisfies $TQ_F^t = 0$. We introduce 11×34 matrix which satisfies

$TU^t = \mathbf{1_{11\times 11}}$.

$$U = \left(\begin{array}{ccccccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{0_{11\times 21}} \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \end{array} \right) . \tag{4.9}$$

Obviously the choice of U matrix is not unique. For a moment we just put all other matrix elements from 14 columns to 34 ones. It is easy to see $\mathrm{U}(1)^7$ charge matrix,

$$V = \left(\begin{array}{c|ccccccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & y_1 & y_2 & y_3 & z_1 & z_2 & w_1 \\ \hline q_1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ q_2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ q_3 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ q_4 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ q_5 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ q_6 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ q_7 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{array} \right) , \tag{4.10}$$

where we can see these charge assignment by remembering (4.4) . For example, because of $x_1 = X_{18}$, the charges of x_1 are $q_1 = 1$ and $q_8 = -1$ but we took only 7 charges($q_8 = -\sum_{i=1}^{7} q_i$). Then 34 homogeneous variables, $p_1, \ldots, p_{34}$ have charges

$VU = Q_D$ by multiplying V and U, given by

$$VU = Q_D = \left(\begin{array}{ccccccccccccc|c} 1 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 1 & 0 & -1 & 1 & -1 & 0 & 0 & -1 & 0 & 1 & \\ -1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & \\ -1 & 1 & 0 & 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & \mathbf{0_{7\times 21}} \\ 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & \end{array} \right) . \tag{4.11}$$

Combining this with the charge matrix Q_F, by including Fayet-Illiopoulos parameters in order to describe the moduli space in a simple form, Q^{total} becomes

$$Q^{\text{total}} = \left(\begin{array}{c|c} & \xi_1 \\ & \xi_2 \\ & \xi_3 \\ Q_D & \xi_4 \\ & \xi_5 \\ & \xi_6 \\ & \xi_7 \\ \hline Q_F & \mathbf{0_{23\times 1}} \end{array} \right) \tag{4.12}$$

where ξ_i correspond to $r_{a,b,c}$ appeared in last section. In an appropriate region of ξ space, we expect that this will give us exactly same description of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ quotient singularity given in (2.13). The 6×10 matrix elements can be obtained from 30×34 matrix elements by doing row operations, the remaining 6×24 vanish and in the matrix 24×24 elements 24 homogeneous coordinates can be expressed in terms of 10 independent ones.

5. Discussion

Note that one of the algebraic equations we have found, $z_1 z_2 z_5 = z_3 z_4$, appears $\mathcal{N} = 1$ $\mathrm{SU}(N) \times \mathrm{SU}(N) \times \mathrm{SU}(N)$ gauge theory on N D3 branes at the singular point in 4 dimensions as pointed out by [14]. The chiral matter fields transform as

$$\begin{aligned} A &= (\square, \bar{\square}, 1)\,, \qquad & B &= (1, \square, \bar{\square})\,, \qquad & C &= (\bar{\square}, 1, \square)\,, \\ \overline{A} &= (\bar{\square}, \square, 1)\,, \qquad & \overline{B} &= (1, \bar{\square}, \square)\,, \qquad & \overline{C} &= (\square, 1, \bar{\square})\,. \end{aligned} \tag{5.1}$$

The minimal set of gauge invariant objects

$$z_1 = |A|^2\,, \qquad z_2 = |B|^2\,, \qquad z_3 = ABC\,, \qquad z_4 = \overline{ABC} \qquad z_5 = |C|^2 \tag{5.2}$$

satisfy the following constraints

$$z_1 z_2 z_5 = z_3 z_4\,. \tag{5.3}$$

This space defines a hypersurface in $\mathbb{C}^5$ and the three lines of singularities meet at the origin. In the present our problem, we can think of the following gauge invariant objects

$$\begin{aligned} z_1 &= |\Phi^1_{111}|^2\,, \\ z_2 &= |\Phi^3_{211}|^2\,, \\ z_3 &= \Phi^1_{111}\Phi^3_{211}\Lambda^{24}_{212}\,, \\ z_4 &= \overline{\Phi^1_{111}\Phi^3_{211}\Lambda^{24}_{212}}\,, \\ z_5 &= |\Lambda^{24}_{212}|^2\,. \end{aligned} \tag{5.4}$$

It is easy to see the gauge invariance by realizing that for the case of the product of two fields the arrow in brane cub model starts and come back its original position and for the triple product of fields they form closed oriented triangles like as the one in superpotential. Of course, they are not independent and are subject to the constraint given by (5.3).

From the toric analysis we have seen previous section, for example, in codimension 2 in the parameter space , we find specific singularity. In order to find the worldvolume theory we take $\eta_3, \eta_4, \eta_5, \eta_6 \gg 0$ by keeping $\eta_1 = \eta_2 = 0$. Let us set the vacuum expectation values as follows

$$\begin{aligned} \Phi^1_{1,1,1} &= \sqrt{\eta_3}\,, \qquad & \Phi^2_{1,2,1} &= \sqrt{\eta_4} \\ \Phi^3_{1,1,2} &= \sqrt{\eta_5}\,, \qquad & \Phi^4_{1,2,2} &= \sqrt{\eta_6} \end{aligned} \tag{5.5}$$

and all others vanish. Then this breaks gauge group down to $\mathrm{U}(N)_{2,1,1} \times \mathrm{U}(N)_{2,2,1} \times \mathrm{U}(N)_{2,1,2} \times \mathrm{U}(N)_{2,2,2}$. By inserting these vevs into the superpotential masses for some of other superfields appear. They can be integrated out by requiring the equation of motion. We finally get the light fields and their charges.

When one is interested in placing M2 branes at the $\mathbb{C}^4/\Gamma$ singularity[3] in the context of AdS/CFT correspondence, one can choose one compact direction in the configuration, shrink it to get a Type IIA picture and then perform T dualities until the singularities have been transformed into branes. One possibility is as follows

KK monoples : $(x^0, x^1, x^2, x^3, x^4, x^5, x^6)$,
KK' monopoles : $(x^0, x^1, x^2, x^3, x^4, x^9, x^{10})$,
KK" monopoles : $(x^0, x^1, x^2, x^5, x^6, x^9, x^{10})$,
M2 branes : (x^0, x^1, x^2).

[3]We thank A.M. Uranga for making this paragraph.

By shrinking x^{10} direction, we obtain D6 branes along the directions $(x^0, x^1, x^2, x^3, x^4, x^5, x^6)$, KK' monopoles along the directions $(x^0, x^1, x^2, x^3, x^4, x^9)$, KK" monopoles $(x^0, x^1, x^2, x^5, x^6, x^9)$, and D2 branes along the directions (x^0, x^1, x^2). T duality along x^4 and x^6 directions lead to D4 branes along the directions $(x^0, x^1, x^2, x^3, x^5)$, NS5 branes along the directions $(x^0, x^1, x^2, x^3, x^4, x^9)$, NS5' branes along the directions $(x^0, x^1, x^2, x^5, x^6, x^9)$, and D4 branes along the directions $(x^0, x^1, x^2, x^4, x^6)$. The last three kinds of branes look like a brane box model. The first kind of brane (the D4 which is related to the KK at the beginning) breaks some of the supersymmetry. Also, the nice symmetry between the different $\mathbb{Z}_k$ factors in the $\mathbb{C}^4/(\mathbb{Z}_k \times \mathbb{Z}_{k'} \times \mathbb{Z}_{k''})$ singularity is not clear in these dual pictures (for instance, above two of the original KK monopoles become NS5 branes, but the other becomes D4 branes). Another compactification [20] gives 3d $\mathcal{N} = 1$ theory where all three kinds of NS5 branes have common x^3 direction by taking the worldvolume of third NS5" branes as $(x^0, x^1, x^2, x^4, x^6, x^8)$ rather than $(x^0, x^1, x^4, x^5, x^6, x^7)$. This theory is non chiral and needs to be studied further. Recently [21] it was observed that the brane box model should be modified by adding brane diamond in order to be brane box model as dual of the blowup of the orbifolded conifold and of the deformed generalized conifold. Although we have not thought about this too much we would expect that this phenomena will occur also in the brane cub model we discussed in section 3.

The toric realization [22] of $\mathbf{P}^3$ has a tetrahedron over each interior point of which there exists 3 torus, which shrinks to a 2 torus at four boundary faces, where it shrinks to a circle at each edge of the tetrahedron, and where it shrinks to a point at each vertex of the tetrahedron. Each face of the tetrahedron with a two torus on top corresponds to a $\mathbf{P}^2$ and each edge with a circle on top corresponds to a $\mathbf{P}^1$. However, the $\mathbf{P}^3$ itself can not be useful, it can be used a local geometry of Calabi-Yau four fold near a singularity. It is known that $N(\mathbf{P}^3)$ appears part of Calabi-Yau four fold compactification where the coordinates of which are represented by (z_1, z_2, z_3, p) corresponding to $\mathbf{P}^3$ and the cotangent direction.

When $\mathbf{P}^3$ is embedded in a Calabi-Yau fourfold, there exists a normal direction corresponding to a line bundle on $\mathbf{P}^3$. The property of $c_1 = 0$ for the fourfold gives that the normal bundle is a canonical line bundle. The normal direction to $\mathbf{P}^3$ can be identified with the space of $(3, 0)$ forms on $\mathbf{P}^3$. Now we have a four dimensional local toric geometry. The extra circle action plays th role of the rotation on the phase of the normal line bundle. The toric realization of $N(\mathbf{P}^3)$ can be viewed as follows: A copy of $\mathbf{P}^3$ is placed as the tetrahadron at the bottom. Each semiinfinite face emanating from any line on it corresponds to the normal direction of $\mathbf{P}^3$ in the Calabi-Yau fourfold. For M theory on a local singularity of a Calabi-Yau fourfold($\mathbf{P}^3$ shrinking inside a Calabi-Yau fourfold), the local model is the canonical bundle over $\mathbf{P}^3$, $N(\mathbf{P}^3)$. By modding out T^3 action corresponding to three circle action on the $\mathbf{P}^3$, there exist 5 dimensional space $N(\mathbf{P}^3)/T^3$ which is trivial except for the loci where a circle action of T^3 has fixed points. The brane realization of $\mathbf{P}^3$ was observed in [22]: There exist four faces of the tetrahedron corresponding to (p, q, r) 4 branes as well as six semiinfinite faces ending on each six edge of tetrahedron, corresponding to (p, q, r) external 4 branes. It would be interesting to study how the toric geometry arise in these brane configuration in detail.

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